

# SYMPLECTIC COBORDISMS AND THE STRONG WEINSTEIN CONJECTURE

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**ABSTRACT.** We study holomorphic spheres in certain symplectic cobordisms and derive information about periodic Reeb orbits in the concave end of these cobordisms from the non-compactness of the relevant moduli spaces. We use this to confirm the strong Weinstein conjecture (predicting the existence of null-homologous Reeb links) for various higher-dimensional contact manifolds, including contact type hypersurfaces in subcritical Stein manifolds and in some cotangent bundles. The quantitative character of this result leads to the definition of a symplectic capacity.

## 1. INTRODUCTION

In a previous paper [18] we studied moduli spaces of holomorphic discs in certain 4-dimensional symplectic cobordisms. As in the classical work of Hofer [21] on the Weinstein conjecture in dimension 3, the non-compactness of these moduli spaces was used to detect periodic Reeb orbits. Moreover, this approach enabled us to give a unified view of many results in 3-dimensional contact topology and 4-dimensional symplectic topology.

In the present paper we extend this work to higher dimensions. Using an idea that can be traced back to McDuff [25], we modify our set-up by constructing a symplectic cap on the convex end of the symplectic cobordism. We can then work with moduli spaces of holomorphic spheres rather than discs, which allows us to invoke a compactness theorem from symplectic field theory [5].

Our main technical result (Theorem 3.1) makes quantitative predictions about periodic Reeb orbits in the concave end of the symplectic cobordisms under consideration. In Corollary 3.3 we rephrase this as a statement about the Weinstein conjecture [35] in the strong version proposed by Abbas et al. [1], which will be recalled in Section 2. We also recover a result of McDuff about a class of symplectic fillings whose boundary is necessarily connected (Theorem 3.4).

In Section 4 we explore various applications of these results. Our main focus is on specific instances of the strong Weinstein conjecture and on quantitative Reeb dynamics. These examples include contact type hypersurfaces in subcritical Stein manifolds (Corollary 4.2) and in cotangent bundles over split manifolds  $Q \times S^1$  (Corollary 4.8). As in our previous paper, the quantitative results give rise to the definition of a symplectic capacity via the periods of Reeb orbits on contact type hypersurfaces.

A typical application is the Weinstein conjecture for subcritically Stein fillable contact manifolds (Corollary 4.3). This result is complementary to that of Albers–Hofer [3] on the Weinstein conjecture for higher-dimensional contact manifolds that

are overtwisted in the sense of Niederkrüger [27] and hence, as shown there, do not admit semipositive strong symplectic fillings. Albers–Hofer study holomorphic discs in trivial symplectic cobordisms only, but it seems likely that this can be extended to the more general cobordisms considered here. For other recent work on the higher-dimensional Weinstein conjecture see [28].

The final two sections contain the proof of the main technical result. In Section 5 we describe a completion of the symplectic cobordism. In Section 6 we introduce certain moduli spaces of holomorphic spheres in this completed cobordism; the non-compactness of these moduli spaces implies the main result.

Part of the motivation for this paper comes from the recent article by Oancea–Viterbo [30] on the topology of symplectic fillings. At the end of the present paper we briefly indicate the relation of our results with their work.

## 2. THE STRONG WEINSTEIN CONJECTURE

Let  $M$  be a closed  $(2n-1)$ -dimensional manifold carrying a (cooriented) contact structure  $\xi$ , i.e. a tangent hyperplane field defined as  $\xi = \ker \alpha$  for some 1-form  $\alpha$  such that  $\alpha \wedge (d\alpha)^{n-1}$  is a volume form on  $M$ . We equip  $M$  with the orientation induced by this volume form; write  $\overline{M}$  for  $M$  with the reversed orientation. The Reeb vector field  $R = R_\alpha$  of  $\alpha$  is defined by the equations  $i_R d\alpha = 0$  and  $\alpha(R) = 1$ .

Paraphrasing Weinstein [35] we say that  $(M, \xi)$  *satisfies the Weinstein conjecture* if for *every* contact form  $\alpha$  defining  $\xi$  the corresponding Reeb vector field  $R_\alpha$  has a closed orbit.

When we speak of a contractible periodic orbit, the period is not required to be the minimal one.

**Definition.** A **Reeb link** for a contact form  $\alpha$  is a collection of periodic orbits of  $R_\alpha$ , not necessarily of minimal period. Its **total action** is the sum of the periods.

In other words, when we speak of a Reeb link we allow the components of this link to be multiply covered. This convention is important for the following definition.

**Definition.** Following Abbas et al. [1] we say that  $(M, \xi)$  *satisfies the strong Weinstein conjecture* if for every  $\alpha$  defining  $\xi$  there exists a *nullhomologous* Reeb link.

Here is a simple example of a whole class of contact manifolds  $(M, \alpha)$  admitting nullhomologous Reeb links. Consider a closed manifold  $B$  with a symplectic form  $\omega$  such that  $\omega/2\pi$  represents an integral cohomology class. Let  $\pi: M \rightarrow B$  be a principal circle bundle of (real) Euler class  $e = -[\omega/2\pi]$ . Then the connection 1-form  $\alpha$  on this bundle with curvature form  $\omega$ , i.e.  $d\alpha = \pi^*\omega$ , is a contact form on  $M$  whose Reeb orbits are the fibres of the  $S^1$ -bundle. When  $B$  is a surface, the Euler class may be regarded as an integer, and the  $|e|$ -fold multiple of the fibre is nullhomologous in  $M$ . Unless  $B$  is a 2-sphere, no multiple of the fibre will be nullhomotopic. This follows from the homotopy exact sequence

$$\dots \longrightarrow \pi_2(B) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow \dots$$

of the fibration, since surfaces of genus at least one are aspherical. In the case  $\dim B \geq 4$ , choose a 2-dimensional integral homology class of  $M$  on which  $e$  evaluates non-trivially. Any such class can be represented by an embedded surface  $\Sigma$ , see [31, Théorème II.27]. Then the  $|e(\Sigma)|$ -fold multiple of the fibre will be homologically trivial in  $\pi^{-1}(\Sigma)$  and, *a fortiori*, in  $M$ .

## 3. THE MAIN TECHNICAL RESULT

Our main theorem answers the strong Weinstein conjecture in the affirmative for contact manifolds  $(M, \alpha)$  that arise as the strong concave boundary of a suitable compact symplectic cobordism  $(W, \omega)$  of dimension  $2n$ , which we equip with the orientation given by the volume form  $\omega^n$ . Throughout this paper it is understood that  $n \geq 2$ .

The convex end of  $(W, \omega)$  is required to contain one connected component (or collection of components)  $S$  of the type we describe next. This class of manifolds includes spheres and ellipsoids with their standard contact form.

**3.1. The boundary component  $S$ .** Let  $(P, \omega_P)$  be a compact, connected  $(2n-2)$ -dimensional symplectic manifold (with boundary) admitting a strictly plurisubharmonic potential, by which we mean the following. We require the existence of an almost complex structure  $J_P$  tamed by  $\omega_P$ , i.e.  $\omega_P(X, J_P X) > 0$  for all non-zero tangent vectors  $X$ , and a smooth function  $\psi_P: P \rightarrow \mathbb{R}$  having the boundary  $\partial P$  as a regular level set and with

$$\omega_P = -d(\psi_P \circ J_P).$$

A straightforward calculation, cf. [17, Lemma 4.11.3], shows that the restriction of a (strictly) plurisubharmonic function to a non-singular holomorphic curve is (strictly) subharmonic, whence the name. This entails the maximum principle for such curves (even in the non-strict case).

Given any point of an almost complex manifold and a holomorphic tangent vector at that point, one can find a local holomorphic curve passing through that point in the given tangent direction [29, Theorem III]. Combined with the maximum principle this implies that  $\psi_P$  attains its maximum on the boundary  $\partial P$  only. After changing  $\psi_P$  by an additive constant we may assume  $\min \psi_P = 0$ .

Equip the product  $C := P \times \mathbb{CP}^1$  with the almost complex structure  $J_C := J_P \oplus i$  and the symplectic form  $\omega_C := \omega_P + \omega_{\text{FS}}$ , where  $\omega_{\text{FS}}$  denotes the Fubini–Study form of total integral  $\pi$  on  $\mathbb{CP}^1$ . Observe that the Kähler manifold  $(\mathbb{CP}^1, \omega_{\text{FS}})$  may be interpreted as the one-point compactification of the open unit disc  $B_1$  in  $\mathbb{C}$  with its standard area form. On  $P \times B_1$  we have the corresponding strictly plurisubharmonic function  $\psi := \psi_P + |z|^2/4$ . We write  $\mathbb{CP}^1 = B_1 \cup \{\infty\}$  and  $P_\infty := P \times \{\infty\} \subset C$ .

The manifold  $S$  is supposed to be a regular level set  $\psi^{-1}(c)$  with  $c$  smaller than both  $\max \psi_P$  and  $1/4$ , or a collection of connected components of such a level set. This choice of  $c$  ensures that  $S$  is a compact hypersurface in  $\text{Int}(P) \times B_1$ . It inherits the contact form

$$\alpha_S := -d\psi \circ J_C|_{TS};$$

the contact structure  $\ker \alpha_S$  is given by the  $J_C$ -invariant sub-bundle of the tangent bundle  $TS$ . The Liouville vector field  $Y$  for  $\omega_C$  defined by  $i_Y \omega_C = -d\psi \circ J_C$  satisfies  $d\psi(Y) > 0$ , so the contact manifold  $(S, \alpha_S)$  is the strong convex boundary of the symplectic manifold  $(\psi^{-1}([0, c]), \omega_C)$ .

**3.2. The symplectic cobordism  $(W, \omega)$ .** Here are the properties we require of the symplectic cobordism  $(W, \omega)$ :

- (C1)  $(W, \omega)$  is compact, connected and  $\pi$ -semipositive (see below for the definition).

(C2) The oriented boundary of  $W$  equals

$$\partial W = \overline{M} \sqcup M_+ \sqcup S,$$

where  $M_+$  is allowed to be empty. The manifolds  $M$ ,  $M_+$  and  $S$  are not required to be connected.

(C3)  $(M, \alpha)$  is the strong concave boundary of  $(W, \omega)$ . By this we mean that there is a Liouville vector field  $Y$  for  $\omega$  defined near  $M \subset W$  and pointing into  $W$  along  $M$  such that  $\alpha = i_Y \omega|_{TM}$ .

(C4) On a neighbourhood of  $M_+ \subset W$  there is an  $\omega$ -tame almost complex structure  $J_+$  relative to which the boundary  $M_+$  is  $J_+$ -convex, i.e. the  $J_+$ -invariant subbundle of  $TM_+$  is a contact structure.

(C5)  $(S, \alpha_S)$  is a strong convex boundary component of  $(W, \omega)$ .

For the definition of  $\pi$ -semipositivity, which is essentially the one given in [26, Definition 6.4.5], recall that a homology class  $A \in H_2(W)$  is said to be *spherical* if it lies in the image of the Hurewicz homomorphism  $\pi_2(W) \rightarrow H_2(W)$ . Write  $c_1$  for the first Chern class of the symplectic manifold  $(W, \omega)$ , defined via any almost complex structure on  $W$  in the contractible space of  $\omega$ -tame structures, and  $c_1(A)$  for the evaluation of this class on  $A$ . By  $\omega(A)$  we denote the evaluation of the de Rham cohomology class  $[\omega] \in H_{\text{dR}}^2(W)$  on  $A$ .

**Definition.** Let  $\kappa$  be a positive real number. A symplectic manifold  $(W, \omega)$  of dimension  $2n$  is  $\kappa$ -**semipositive** if any spherical class  $A \in H_2(W)$  with  $0 < \omega(A) < \kappa$  and  $c_1(A) \geq 3 - n$  satisfies  $c_1(A) \geq 0$ .

Note that this condition is automatically satisfied for symplectic manifolds of dimension at most 6.

One particular case of interest to us is the one where, in addition to conditions (C1) to (C5), the symplectic form  $\omega$  is an exact form  $\omega = d\lambda$  with  $\lambda|_{TM} = \alpha$ . We shall refer to this as the *Liouville case*, since the conditions are those of a concave boundary in a Liouville cobordism, cf. [18]. In this case, too,  $\kappa$ -semipositivity is automatic.

**3.3. The main theorem.** We can now formulate our main result.

**Theorem 3.1.** *Given a symplectic cobordism  $(W, \omega)$  satisfying conditions (C1) to (C5), there exists a nullhomologous Reeb link in its concave end  $(M, \alpha)$  of total action smaller than  $\pi$ . In the Liouville case there is in fact a contractible Reeb orbit of period smaller than  $\pi$ .*

**Remark 3.2.** The choice  $\kappa = \pi$  in (C1) is made purely for notational convenience. In the general case, one would have to replace the Fubini–Study form  $\omega_{\text{FS}}$  by  $(\kappa/\pi)\omega_{\text{FS}}$ , and the open unit disc  $B_1$  by a disc of radius  $\sqrt{\kappa/\pi}$ . The action of the Reeb link predicted by our theorem would then be smaller than  $\kappa$ .

A neighbourhood of  $M \subset (W, \omega)$  looks like a neighbourhood of  $\{0\} \times M$  in the half-symplectisation  $([0, \infty) \times M, d(e^s \alpha))$ . This allows us to define a symplectic form  $\omega_-$  on the manifold

$$(-\infty, 0] \times M \cup_M W$$

by

$$\omega_- := \begin{cases} \omega & \text{on } W, \\ d(e^s \alpha) & \text{on } (-\infty, 0] \times M. \end{cases}$$

Any contact form defining the cooriented contact structure  $\xi := \ker \alpha$  can be written as  $e^h \alpha$  for some smooth function  $h: M \rightarrow \mathbb{R}$ . Rescaling this by a constant function (which does not change the Reeb dynamics qualitatively) we may assume that  $h$  takes negative values only. Replacing  $M$  by

$$\{(h(x), x) \in (-\infty, 0] \times M : x \in M\} \subset (-\infty, 0] \times M \cup_M W$$

we obtain a cobordism as in Theorem 3.1, with concave boundary  $(M, e^h \alpha)$ .

In the Liouville case we can define the collar of  $M$  in  $W$  via the Liouville vector field  $Y \equiv \partial_s$  given by  $i_Y \omega = \lambda$ . Then both  $\lambda$  and  $e^s \alpha$  are  $Y$ -invariant and evaluate to zero on  $Y$ ; since they coincide on  $TM$ , they coincide near  $M$ . In other words,  $e^s \alpha$  defines an extension of the primitive  $\lambda$  to  $(-\infty, 0] \times M$ .

These considerations lead to the following corollary.

**Corollary 3.3.** *Given a symplectic cobordism  $(W, \omega)$  satisfying conditions (C1) to (C5), the contact manifold  $(M, \xi)$  satisfies the strong Weinstein conjecture. In the Liouville case, any contact form defining  $\xi$  has a contractible periodic Reeb orbit.*  $\square$

**Examples.** (1) Let  $(M, \xi)$  be a closed contact manifold of dimension 3 or 5 occurring as the concave end of a strong symplectic cobordism whose convex end is  $S^3$  or  $S^5$ , respectively, with its standard contact structure  $\xi_{\text{st}}$ . Then  $(M, \xi)$  satisfies the strong Weinstein conjecture.

(2) If  $(M, \xi)$  with  $\dim M = 2n - 1$  is Liouville cobordant to  $(S^{2n-1}, \xi_{\text{st}})$ , e.g. if  $(S^{2n-1}, \xi_{\text{st}})$  can be obtained from  $(M, \xi)$  by contact surgery, then any contact form defining  $\xi$  has a contractible closed Reeb orbit.

Related cobordism-theoretic arguments allow one to reprove the result of Abbas et al. [1] that planar contact structures on 3-manifolds satisfy the strong Weinstein conjecture. Our proof of Theorem 3.1 rests on the construction of a symplectic cap, see Section 5.1 below. A cap in the case of planar contact structures has been constructed by Etnyre [13]. Details of the ensuing holomorphic curves argument can be found in [10].

The methods for proving Theorem 3.1 also yield the following result. On the face of it, this is stronger than a result of McDuff [25, Theorem 1.4], but her proof actually yields the result we formulate here. In fact, the proof we give in Section 6.5 below is essentially hers, except that we paraphrase it in the more sophisticated language of [26].

**Theorem 3.4** (McDuff). *Let  $(W, \omega)$  be a symplectic cobordism satisfying conditions (C1) to (C5), but now with the additional assumption that  $M$  be empty, i.e. there is no concave boundary component. Then  $M_+$  is likewise empty.*

#### 4. APPLICATIONS

Before we turn to the proof of Theorem 3.1, we describe a number of applications. Some are parallel to the 4-dimensional applications of the ‘ball theorem’ proved in [18]; we shall be brief in the discussion of those.

**4.1. Reeb dynamics.** In [32] Viterbo proved the existence of closed characteristics on compact contact type hypersurfaces in  $\mathbb{R}^{2n}$  with its standard symplectic structure. In [34, Theorem 4.4] he extended this to contact type hypersurfaces in subcritical Stein manifolds; an alternative proof was given by Frauenfelder–Schlenk [16,

Corollary 3]. Our main theorem allows us to prove the strong Weinstein conjecture in these situations. First we recall the basic definitions.

**Definition.** A hypersurface  $M$  in a symplectic manifold  $(V, \omega)$  is said to be of **contact type** if there is a Liouville vector field  $Y$  for  $\omega$  defined near and transverse to  $M$ . The hypersurface is said to be of **restricted contact type** if  $Y$  is defined on all of  $V$ .

It will be understood that a contact type hypersurface  $M$  is equipped with the contact form  $i_Y \omega|_{TM}$ . If  $Y$  and hence the primitive  $i_Y \omega$  for  $\omega$  is globally defined, this places us in the Liouville case of our main theorem. Recall our convention from Section 2: when we say that  $M$  satisfies the Weinstein conjecture, we mean that *every* contact form defining the contact structure  $\ker(i_Y \omega|_{TM})$  has a closed Reeb orbit. Periodic Reeb orbits of the specific contact form  $i_Y \omega|_{TM}$  will be referred to as **closed characteristics**.

**Definition.** A **Stein manifold** is a complex manifold  $(V, J)$  admitting a proper holomorphic embedding into some  $\mathbb{C}^N$ . Then  $(V, J)$  admits an exhausting (i.e. bounded from below and proper) strictly plurisubharmonic function  $\psi$ , e.g. the restriction of the function  $\sum_{k=1}^N |z_k|^2$  on  $\mathbb{C}^N$ . The 2-form

$$\omega_\psi := -d(d\psi \circ J)$$

is then a symplectic form on  $V$ . By [12, Theorem 1.4.A], any other exhausting strictly plurisubharmonic function on  $(V, J)$  gives rise to a symplectomorphic copy of  $(V, \omega_\psi)$ .

A **Stein domain** is a regular sub-level set  $\{\psi \leq c\}$ ; this is also called a **Stein filling** of the level set  $\psi^{-1}(c)$  with contact structure given by the  $J$ -invariant sub-bundle of its tangent bundle.

If  $\psi$  is also a Morse function, then the index of any of its critical points is at most equal to  $(\dim_{\mathbb{R}} V)/2$ , cf. Remark 5.1 below. A Stein manifold or domain is called **subcritical** if  $\psi$  is Morse with all critical points of Morse index strictly smaller than  $(\dim_{\mathbb{R}} V)/2$ .

**Remark 4.1.** Stein fillability of a contact manifold  $(M, \xi)$  can also be defined by requiring the existence of a compact complex manifold  $V$  with boundary  $M$ , admitting a strictly plurisubharmonic function  $\psi$  for which  $M$  is a regular level set  $\psi^{-1}(c)$ . The interior  $\text{Int}(V)$  then admits an exhausting strictly plurisubharmonic function (with the same level sets as  $\psi$ ): simply replace  $\psi$  by  $h \circ \psi$  with  $h: (-\infty, c) \rightarrow \mathbb{R}$  convex and strictly increasing, with  $h(x) \rightarrow \infty$  as  $x \rightarrow c$ . By Grauert's famous theorem [19],  $\text{Int}(V)$  is then again a Stein manifold. Moreover, thanks to Gray stability, a level set sufficiently close to  $\infty$  will be a contactomorphic copy of  $(M, \xi)$ .

**Corollary 4.2.** *Any smooth compact hypersurface of contact type (with respect to some symplectic form  $\omega_\psi$ ) in a subcritical Stein manifold satisfies the strong Weinstein conjecture. If the hypersurface is of restricted contact type, every contact form defining the induced contact structure has a contractible periodic Reeb orbit.*

*Proof.* By a result of Cieliebak [6], cf. [7], any subcritical Stein manifold is symplectomorphic to a split one  $(V \times \mathbb{C}, J_V \oplus i)$ , where  $(V, J_V)$  is some Stein manifold. So we have a strictly plurisubharmonic function  $\psi_V$  on  $V$ , and  $\psi := \psi_V + |z|^2/4$  is a strictly plurisubharmonic function on  $V \times \mathbb{C}$ .

Thus, we are dealing with a compact hypersurface  $(M, \xi)$  of (restricted) contact type in  $(V \times \mathbb{C}, \omega_\psi)$ . Without loss of generality we may take  $M$  to be connected. Since  $V \times \mathbb{C}$  has trivial homology in codimension 1,  $M$  separates  $V \times \mathbb{C}$  into a bounded and an unbounded part. Choose a regular level set  $\psi^{-1}(c)$  containing  $M$  in the interior, and write  $W$  for the part between  $M$  and  $\psi^{-1}(c)$ . The Liouville vector field  $Y$  near  $M$  (or on all of  $W$  in the case of restricted contact type) points into  $W$  along  $M$ , otherwise Theorem 3.4 would be violated. Hence Corollary 3.3 applies to this symplectic cobordism  $W$ . (For this qualitative result, it is irrelevant that we have replaced  $B_1$  by  $\mathbb{C}$ .)  $\square$

As this proof shows, the corollary is close in spirit to the work of Floer et al. [14], where the existence of closed characteristics is proved in split symplectic manifolds  $P \times \mathbb{C}^l$  with  $P$  closed and  $\pi_2(P) = 0$ .

The following corollary extends earlier work of Andenmatten [4, Theorem 1.4] and Yau [36], who impose additional homological conditions on the Stein filling.

**Corollary 4.3.** *If  $(M, \xi)$  is a subcritically Stein fillable contact manifold, any contact form defining  $\xi$  has a contractible periodic Reeb orbit.*

*Proof.* By assumption,  $(M, \xi)$  is a level set  $\psi^{-1}(c)$  of an exhausting strictly plurisubharmonic Morse function  $\psi$  on a Stein manifold  $(V, J)$ . A contact form defining  $\xi$ , the  $J$ -invariant sub-bundle of  $TM$ , is given by the restriction of the global primitive  $-\mathrm{d}\psi \circ J$  of  $\omega_\psi$ . So we are in the restricted contact type case of the preceding corollary.  $\square$

**Remark 4.4.** The Floer-homological methods of Viterbo [34] and Frauenfelder–Schlenk [16] produce a closed Reeb orbit contractible in the ambient manifold. In the situation of Corollary 4.3,  $M$  is disjoint from the isotropic skeleton of the Stein filling, and by general position a closed orbit contractible in the subcritical filling is also contractible in  $M$  itself. So that last corollary can alternatively be derived from their result.

**4.2. Capacities and non-squeezing.** Given a closed manifold  $M$  with contact form  $\alpha$  we write  $\inf(\alpha)$  for the infimum of all positive periods of closed orbits of the Reeb vector field  $R_\alpha$ . When there are no closed Reeb orbits, we have  $\inf(\alpha) = \infty$ , otherwise an Arzelà–Ascoli type argument as in [23, p. 109] shows that  $\inf(\alpha)$  is a minimum, and in particular positive; the latter is also a simple consequence of the flow box theorem.

Let  $(V, \omega)$  be a  $2n$ -dimensional symplectic manifold. The manifold  $V$  is allowed to be non-compact or disconnected. It may also have non-empty boundary, in which case  $V$  should be replaced by  $\mathrm{Int}(V)$  in the following definition of a symplectic invariant of  $(V, \omega)$ :

$$c(V, \omega) := \sup_{(M, \alpha)} \{ \inf(\alpha) \mid \exists \text{ contact type embedding } (M, \alpha) \hookrightarrow (V, \omega) \}.$$

Here the supremum is taken over all closed, but not necessarily connected, contact manifolds  $(M, \alpha)$  of dimension  $2n - 1$ . By a contact type embedding  $j: (M, \alpha) \hookrightarrow (V, \omega)$  we mean that there is a Liouville vector field  $Y$  for  $\omega$  defined near  $j(M)$  such that  $j^*(i_Y \omega) = \alpha$ .

In  $\mathbb{R}^{2n} \equiv \mathbb{C}^n$  with its standard symplectic form  $\omega_{\mathrm{st}} = (i/2) \mathrm{d}z \wedge \mathrm{d}\bar{z}$  write  $B_r^{2n}$  for the open  $2n$ -ball of radius  $r$  and  $Z_r = \mathbb{C}^{n-1} \times B_r^2$  for the cylinder over the open 2-ball of radius  $r$ . For  $r = 1$  we simply write  $B$  and  $Z$ , respectively.

**Theorem 4.5.** *The invariant  $c(V, \omega)$  is a symplectic capacity, i.e. it satisfies the following axioms:*

**Monotonicity:** *If there exists a symplectic embedding  $(V, \omega) \hookrightarrow (V', \omega')$ , then  $c(V, \omega) \leq c(V', \omega')$ .*

**Conformality:** *For any  $a \in \mathbb{R}^+$  we have  $c(V, a\omega) = a c(V, \omega)$ .*

**Normalisation:**  $c(B) = c(Z) = \pi$ .

*Proof.* Monotonicity and conformality are obvious from the definition. The  $(2n-1)$ -sphere of radius  $r < 1$  with its standard contact form has all Reeb orbits closed of period  $\pi r^2$ , and it admits a contact type embedding into both  $B$  and  $Z$ ; cf. [18], where this is computed explicitly in the 4-dimensional case. This implies that  $c(B)$  and  $c(Z)$  are bounded from below by  $\pi$ .

If  $j$  is a contact type embedding of some  $(2n-1)$ -dimensional contact manifold  $(M, \alpha)$  into  $B$  or  $Z$ , then the image  $j(M)$  is contained in the interior of an ellipsoid

$$E = E_\varepsilon(b, \dots, b, 1) := \left\{ \sum_{k=1}^{n-1} \frac{|z_k|^2}{b^2} + |z_n|^2 \leq 1 - \varepsilon \right\}$$

for  $b > 0$  sufficiently large and  $\varepsilon > 0$  sufficiently small. Now our main theorem applies to the symplectic cobordism given by the region in  $(Z, \omega_{\text{st}})$  between  $j(M)$  and  $\partial E$ .  $\square$

More generally, this argument shows that if  $(P, \omega_P)$  is a connected but not necessarily compact  $(2n-2)$ -dimensional symplectic manifold admitting a plurisubharmonic potential  $\psi_P$  (with the boundary as a regular level set in the compact case), then  $c(P \times B_1, \omega_P + dx \wedge dy) \leq \pi$ . It is not difficult to give examples where equality holds, e.g. if  $P$  is a cotangent bundle or a Stein manifold of finite type.

As an immediate consequence of Theorem 4.5, we have Gromov's non-squeezing theorem [20, p. 310]:

**Corollary 4.6** (Gromov). *There is a symplectic embedding  $B_r^{2n} \hookrightarrow Z_R$  if and only if  $r \leq R$ .*  $\square$

Our main theorem allows us to define other capacities in a similar fashion. One option is to take the infimum over the total action of null-homologous Reeb links. A further possibility, for exact symplectic manifolds, is to work with restricted contact type embeddings and then to take the infimum over contractible Reeb orbits, see [18]. A comprehensive survey on symplectic capacities can be found in [8].

In the case of exact symplectic manifolds  $(V, d\lambda)$ , there are in fact two sensible ways to introduce a capacity. One, carried out in the 4-dimensional setting in [18], is to consider restricted contact type embeddings for the given primitive  $\lambda$ , i.e. embeddings  $j: (M, \alpha) \rightarrow (V, d\lambda)$  with  $j^*\lambda = \alpha$ . This leads to an invariant on the set of exact symplectic manifolds with given primitive. Alternatively, and more in the spirit of an Ekeland–Hofer capacity [11], one can define a capacity exclusively for subsets  $U$  of a given exact symplectic manifold  $(V, \omega = d\lambda)$ . This can be done via restricted contact type embeddings  $j: (M, \alpha) \rightarrow (V, \omega)$ , i.e. it is only required that there be *some* global primitive  $\lambda_j$  for  $\omega$ , depending on the embedding  $j$ , with  $j^*\lambda_j = \alpha$ , but in addition the condition  $j(M) \subset U$  is imposed.

All these capacities have the same normalisation constants, but we do not know if they are different, in general.



**4.3. Quantitative Reeb dynamics.** By way of an example, we show that our capacity can be used to recover a result of Frauenfelder et al. [15, Remark 1.13.3]. We improve the constant in their result by appealing to classical geometry.

**Corollary 4.7.** *Let  $(M, \alpha) \subset (\mathbb{R}^{2n}, \omega_{\text{st}})$  be a compact hypersurface of contact type. Then  $\inf(\alpha) \leq (n/(2n+1))\pi(\text{diam}(M))^2$ .*

*Proof.* Since the symplectic form  $\omega_{\text{st}}$  is translation-invariant, we have a contact type embedding of  $(M, \alpha)$  into  $B_r^{2n}$  for any  $r$  greater than the circumradius of  $M$ , which by [24] is at most equal to  $\sqrt{n/(2n+1)} \text{diam}(M)$ . This bound is optimal; it is attained for the regular  $2n$ -simplex. Hence

$$\inf(\alpha) \leq c(B_r^{2n}) = \pi r^2 \text{ for any } r > \sqrt{n/(2n+1)} \text{diam}(M). \quad \square$$

In the case of restricted contact type hypersurfaces, one obtains the same quantitative estimate on *contractible* closed Reeb orbits, cf. [18].

**4.4. Cotangent bundles.** The Weinstein conjecture for contact type hypersurfaces in a cotangent bundle  $T^*L$  (with its canonical symplectic structure) is of particular interest, since this includes the question of closed characteristics on energy surfaces in classical mechanical systems. The solution to the existence question in this classical case is described in [23, Chapter 4.4]. Hofer–Viterbo [22] proved the existence of closed characteristics on contact type hypersurfaces in  $T^*L$  enclosing the zero section. Viterbo [33, Theorem 3.1], [34, p. 1020] covers the case where the fundamental group of  $L$  is finite.

The following corollary provides new instances of the Weinstein conjecture in cotangent bundles, and in fact gives the strong version.

**Corollary 4.8.** *The strong Weinstein conjecture holds for closed contact type hypersurfaces in  $T^*(Q \times S^1)$ , where  $Q$  is any closed manifold.*

*Proof.* Let  $(M, \xi) \subset T^*(Q \times S^1) = T^*Q \times T^*S^1$  be a closed hypersurface of contact type. We want to show that we can realise  $(M, \xi)$  as a hypersurface of contact type in a symplectic manifold of the form  $P \times \mathbb{C}$  with  $P$  as in Section 3.1. The cobordism  $W$  will then be defined by the part of  $P \times \mathbb{C}$  between  $M$  and a sufficiently high level set  $S$  of the strictly plurisubharmonic potential on  $P \times \mathbb{C}$ . Then we choose  $R$  large enough such that  $S \subset P \times B_R$  and apply Theorem 3.1. For the qualitative statement of the corollary it is irrelevant that we need to replace  $B_1$  in the original formulation of the main theorem by  $B_R$ .

The canonical symplectic form on  $T^*S^1 \cong \mathbb{R} \times S^1$  is given by  $ds \wedge d\theta$ . The compact hypersurface  $M$  lies in  $T^*Q \times \{s > -a/2\} \subset T^*Q \times T^*S^1$  for  $a > 0$  large enough, and a symplectic embedding of  $\{s > -a/2\} \subset T^*S^1$  into  $(\mathbb{C}, \omega_{\text{st}} = r dr \wedge d\theta)$  is given by  $(s, \theta) \mapsto \sqrt{2s+a} e^{i\theta}$ .

In order to finish the proof, we need to equip  $T^*Q$  with an almost complex structure and an exhausting strictly plurisubharmonic function. In [27, Appendix B] it is explained how this can be done, starting from a Riemannian metric on  $Q$ .  $\square$

**Remark 4.9.** Since the symplectic embedding  $\{s > -a/2\} \hookrightarrow \mathbb{C}$  is not surjective, the image of a hypersurface of restricted contact type will only be of non-restricted contact type, in general. So our argument does not allow us to make any statement about contractible periodic Reeb orbits.

Notice that a closed contact type hypersurface in a symplectic manifold of the form  $P \times \mathbb{C}$  is displaceable. In this situation, a result of Frauenfelder–Schlenk [16, Theorem 3] predicts the existence of a closed characteristic.

**4.5. Separating hypersurfaces.** In [2], Albers et al. collect conditions on a 3-dimensional contact manifold that prevent the existence of non-separating contact type embeddings into any 4-dimensional symplectic manifold. Our main theorem and its consequences for the quantitative Reeb dynamics allow us to make such statements in higher dimensions, e.g. for spheres and ellipsoids. However, as we need to control the Reeb dynamics, we need to fix the induced contact form on the hypersurface.

Here is a simple example, pointed out to us by Max Dörner. Consider the standard Liouville form  $\lambda_{\text{st}} = (\mathbf{x} \, d\mathbf{y} - \mathbf{y} \, d\mathbf{x})/2$  on  $\mathbb{R}^{2n}$ , and write  $\alpha_r$  for its restriction to the tangent bundle of the sphere  $S_r := S_r^{2n-1}$  of radius  $r$ .

Recall from Section 4.2 that when we speak of a contact type embedding of  $(S_r, \alpha_r)$  into a symplectic manifold  $(V, \omega)$ , the contact form  $\alpha_r$  being given *a priori*, we mean that there is a Liouville vector field  $Y$  for  $\omega$  defined near and transverse to  $S_r \subset V$  with  $i_Y \omega|_{TS_r} = \alpha_r$ .

**Proposition 4.10.** *Any contact type embedding of  $(S_r, \alpha_r)$  into a closed  $\kappa$ -semi-positive symplectic manifold  $(V, \omega)$  with  $\pi r^2 \leq \kappa$  is separating.*

*Proof.* Suppose we have a non-separating contact type embedding of  $(S_r, \alpha_r)$  into  $(V, \omega)$ . Then a neighbourhood of  $S_r \subset (V, \omega)$  looks like a neighbourhood of the sphere of radius  $r$  in  $(\mathbb{R}^{2n}, \omega_{\text{st}} = d\lambda_{\text{st}})$ . Remove an open tubular neighbourhood around  $S_r \subset V$  corresponding to the shell between  $S_{r-\varepsilon}$  and  $S_{r+\varepsilon}$  in the euclidean model, where  $\varepsilon > 0$  has been chosen sufficiently small. This defines a symplectic cobordism  $(W, \omega)$  from  $(S_{r+\varepsilon}, \alpha_{r+\varepsilon})$  to  $(S_{r-\varepsilon}, \alpha_{r-\varepsilon})$  satisfying the assumptions of Theorem 3.1, where in (C1) we replace  $\pi$ -semipositivity by  $\pi r^2$ -semipositivity. This tells us that there should be a Reeb link in  $(S_{r+\varepsilon}, \alpha_{r+\varepsilon})$  of total action less than  $\pi r^2$ . But all the simple Reeb orbits of  $(S_r, \alpha_r)$  are closed of period  $\pi r^2$ , cf. [18]. This contradiction proves the proposition.  $\square$

**Remark 4.11.** The same argument applies to any ellipsoid whose minimal half-axis satisfies the inequality in the proposition.

## 5. COMPLETING THE SYMPLECTIC COBORDISM

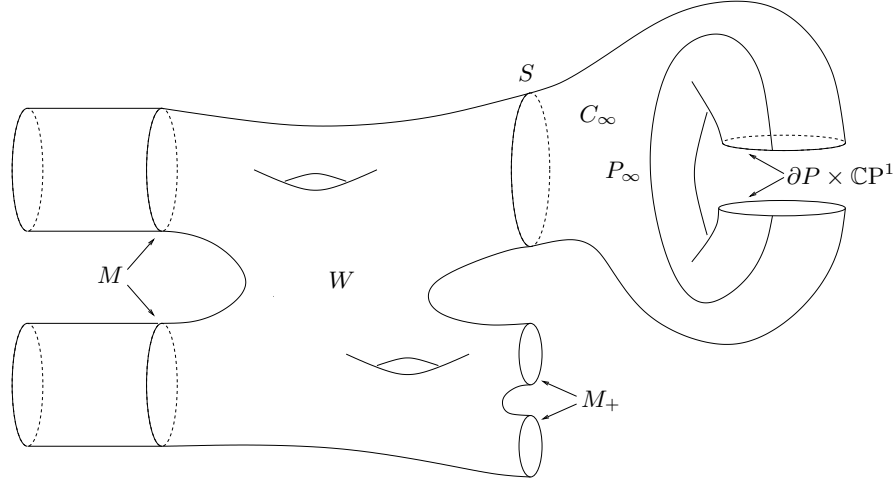
We now begin with the preparations for the proof of the main theorem. We define a ‘completion’ of our symplectic cobordism  $W$  which contains holomorphic spheres, and we describe some simple properties of these holomorphic spheres.

**5.1. The symplectic cap.** The initial step in the proof of Theorem 3.1 is analogous to the arguments of McDuff in [25]. We complete  $W$  by attaching a negative half-symplectisation along  $M$ , as in Section 3.3, and a (perforated) symplectic cap  $(C_\infty, \omega_C)$  along  $S$ , where  $C_\infty$  is the closure of the component of  $C \setminus S$  containing  $P_\infty$  (for the notation cf. Section 3.1).

By gluing the convex boundary  $(S, \alpha_S)$  of  $(W, \omega)$  and the concave boundary  $(S, \alpha_S)$  of  $(C_\infty, \omega_C)$  we obtain the symplectic manifold

$$(\widetilde{W}, \widetilde{\omega}) := ((-\infty, 0] \times M \cup_M W, \omega_-) \cup_S (C_\infty, \omega_C);$$

see Figure 1.

FIGURE 1. The symplectic manifold  $(\widetilde{W}, \tilde{\omega})$ .

**Remark 5.1.** Notice that the Morse index of a non-degenerate critical point of a plurisubharmonic function is at most half the dimension of the almost complex manifold. Otherwise the negative definite subspace of the Hessian at the critical point would contain a complex line, and by [29] we could then find a local holomorphic curve tangent to that line. Such a curve would violate the maximum principle. It follows that if  $\psi_P$  is a Morse function, then the  $(2n - 2)$ -dimensional manifold  $P$  has the homotopy type of a complex of dimension at most  $n - 1$ . The homology exact sequence of the pair  $(P, \partial P)$  then shows that  $\partial P$  is connected when  $n \geq 3$ .

By a  $C^2$ -small perturbation compactly supported in  $\text{Int}(P)$  we can turn any given  $\psi_P$  into a Morse strictly plurisubharmonic function, so this topological conclusion about  $\partial P$  always holds in our set-up. The apparently disconnected  $\partial P$  in Figure 1 is an artefact of the lack of dimensions.

The same argument applied to the connected components of  $\psi^{-1}([0, c])$  shows that all these components have a connected boundary. This means that each component of the level set  $\psi^{-1}(c)$ , and hence in particular  $S$  (which is a collection of such components) is separating.

For more about plurisubharmonic functions on almost complex manifolds see [7].

**5.2. The almost complex structure on  $\widetilde{W}$ .** On the symplectic manifold  $(\widetilde{W}, \tilde{\omega})$  we choose an almost complex structure  $J$  tamed by  $\tilde{\omega}$ , subject to the following conditions:

- (J1) On  $C_\infty \subset C$ , the almost complex structure  $J$  equals the split structure  $J_P \oplus i$ .
- (J2) On the cylindrical end  $(-\infty, 0] \times M$ , the almost complex structure  $J$  is cylindrical and symmetric in the sense of [5, p. 802, 807], i.e. it preserves  $\xi = \ker \alpha$  and satisfies  $J\partial_s = R_\alpha$ .
- (J3) On a neighbourhood of  $M_+$ , the almost complex structure  $J$  equals  $J_+$  (cf. (C4)).

- (J4) Outside the regions described in (J1) to (J3), the almost complex structure is chosen so as to satisfy certain genericity assumptions that will be described later.

**5.3. Holomorphic spheres in  $(\widetilde{W}, J)$ .** Before defining and studying the moduli space of holomorphic spheres in  $(\widetilde{W}, J)$  representing a certain homology class, we collect some information about more general holomorphic spheres that will be relevant in the bubbling-off analysis.

The  $J$ -convexity of  $M_+$  allows one to write  $M_+$  as a level set of a strictly plurisubharmonic function defined in some collar neighbourhood  $U_+$  of  $M_+$ , cf. [18, Remark 4.3]. Then the maximum principle holds for  $J$ -holomorphic curves in  $U_+$ .

Recall from Section 3.1 that  $S$  is a collection of components of a regular level set  $\psi^{-1}(c)$ . Define a closed neighbourhood of  $\partial P \times \mathbb{CP}^1$  by

$$U_\partial := \{p \in P : \psi_P(p) \geq c\} \times \mathbb{CP}^1 \subset C_\infty.$$

This neighbourhood is obviously foliated by holomorphic spheres  $\{p\} \times \mathbb{CP}^1$ .

**Lemma 5.2.** *Let  $u : \mathbb{CP}^1 \rightarrow \widetilde{W}$  be a smooth non-constant  $J$ -holomorphic sphere.*

- (i) *If  $u(\mathbb{CP}^1) \cap C_\infty \neq \emptyset$ , then  $u(\mathbb{CP}^1) \cap P_\infty \neq \emptyset$ .*
- (ii) *If  $u(\mathbb{CP}^1) \cap U_\partial \neq \emptyset$ , then  $u(\mathbb{CP}^1) \subset U_\partial$  and  $u$  is of the form  $z \mapsto (p, v(z))$  with some holomorphic branched covering  $v$  of  $\mathbb{CP}^1$  by itself.*
- (iii) *If  $u(\mathbb{CP}^1) \subset C_\infty$ , then  $u$  is one of the spheres in (ii).*
- (iv)  *$u(\mathbb{CP}^1) \cap U_+ = \emptyset$ .*

*Proof.* (i) Suppose  $u$  is a holomorphic sphere intersecting  $C_\infty$  but not  $P_\infty$ . Then the strictly plurisubharmonic function  $\psi$  is defined on  $u(\mathbb{CP}^1) \cap C_\infty$  and attains a maximum in the interior, forcing  $u(\mathbb{CP}^1) \cap C_\infty$  to lie in a level set of  $\psi$ . So either  $u(\mathbb{CP}^1)$  is completely contained in the exact symplectic manifold  $C_\infty \setminus P_\infty$ , or  $u$  is contained in  $\widetilde{W} \setminus \text{Int}(C_\infty)$  and touches the convex boundary  $S$  of that manifold. Either alternative forces  $u$  to be constant.

(ii) On the preimage of  $U_\partial$  we can write  $u$  in the form  $u(z) = (u_1(z), u_2(z)) \in P \times \mathbb{CP}^1$  with  $u_1$  a  $J_P$ -holomorphic and  $u_2$  a holomorphic function. Thanks to the strictly plurisubharmonic function  $\psi_P$  on  $P$ , the maximum principle applies to  $u_1$ , so as in (i) we argue that  $u$  must be globally of the form  $z \mapsto (p, u_2(z))$ . The non-constant holomorphic map  $u_2$  is a branched covering  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ .

(iii) A sphere  $u$  with image contained in  $C_\infty$  can be globally written as in (ii), that is,  $u(z) = (p, u_2(z))$ . If  $\psi_P(p) \geq c$ , then this is one of the spheres from (ii). If  $\psi_P(p) < c$ , the point  $(p, 0) \in P \times B_1 \subset P \times \mathbb{CP}^1$  does not lie in  $C_\infty$ , so  $u_2$  is not surjective, and hence constant.

(iv) If  $u$  is a holomorphic sphere intersecting  $U_+$ , then with the strictly plurisubharmonic function defined on  $U_+$  we argue as in (i) that  $u$  must be constant.  $\square$

**Remark 5.3.** If  $M_+$  is only weakly  $J$ -convex, which means that it is the level set of a (non-strictly) plurisubharmonic function, then the maximum principle still applies, but there can be non-constant holomorphic spheres entirely contained in a level set. However, such spheres cannot occur in a bubble tree arising as the limit of spheres in the moduli space considered in the next section, because one sphere in such a bubble tree, as we shall see, always intersects  $P_\infty$ , so at least one sphere

would touch a level set but not be entirely contained in it. So the results of the present paper remain valid under this weaker assumption on  $M_+$ .

## 6. THE MODULI SPACE OF HOLOMORPHIC SPHERES

In this section we define the relevant moduli spaces of holomorphic spheres in the completed symplectic cobordism. These moduli spaces will be shown to be non-compact, which then leads to a proof of Theorems 3.1 and 3.4.

**6.1. Spheres in a fixed homology class.** Fix a point  $*$  in  $\partial P$ . Then

$$F := \{*\} \times \mathbb{CP}^1 \subset \widetilde{W}$$

is a holomorphic sphere in  $(\widetilde{W}, J)$ . Write  $\widetilde{\mathcal{M}}$  for the moduli space of smooth  $J$ -holomorphic spheres  $u: \mathbb{CP}^1 \rightarrow \widetilde{W}$  that represent the class  $[F] \in H_2(\widetilde{W})$ .

The intersection number of the classes  $[F] \in H_2(\widetilde{W})$  and  $[P_\infty] \in H_{2n-2}(\widetilde{W}, \partial\widetilde{W})$  equals 1, so with Lemma 5.2 we see that any holomorphic sphere in the class  $[F]$  that touches  $U_\partial$  or is completely contained in  $C_\infty$  must be of the form  $z \mapsto (p, \phi(z))$  with  $\psi_P(p) \geq c$  and  $\phi$  an automorphism of  $\mathbb{CP}^1$ .

**Proposition 6.1.** *For a generic choice of  $J$ , the moduli space  $\widetilde{\mathcal{M}}$  is a smooth manifold (with boundary) of dimension  $2n + 4$ .*

*Proof.* The observation before the proposition tells us that near its boundary the moduli space  $\widetilde{\mathcal{M}}$  looks like a neighbourhood of  $\partial P \subset P$  crossed with the 6-dimensional automorphism group  $\text{Aut}(\mathbb{CP}^1)$ , i.e. like a manifold with boundary of the claimed dimension. So for all practical purposes we can apply transversality arguments as if  $\widetilde{\mathcal{M}}$  had no boundary. A further consequence of the homological intersection of  $[F]$  and  $[P_\infty]$  being equal to 1 is that all spheres in the class  $[F]$  will be simple (i.e. not multiply covered).

The moduli space  $\widetilde{\mathcal{M}}$  will be a manifold provided we can choose  $J$  to be regular in the sense of [26, Definition 3.1.4], i.e. such that the linearised Cauchy–Riemann operator  $D_u$  is surjective for each  $u \in \widetilde{\mathcal{M}}$ .

As to (J1), regularity for spheres contained in  $C_\infty$  follows from their explicit description given before the proposition, cf. [26, Corollary 3.3.5].

In the regions where the choice of  $J$  is prescribed by conditions (J2) and (J3), the maximum principle applies, so no non-constant holomorphic sphere can lie entirely in one of these regions. By [26, Remark 3.2.3], the freedom of choosing  $J$  in the complementary region then suffices to achieve regularity for all holomorphic spheres in  $\widetilde{W}$ , cf. [18, Remark 4.1.(2)].

By [26, Theorem 3.1.5], the dimension of the manifold  $\widetilde{\mathcal{M}}$  is given by  $2n + 2c_1([F])$ . Since the normal bundle of  $F$  in the product manifold  $P \times \mathbb{CP}^1$  is trivial (as a complex bundle),  $c_1([F])$  equals the Euler characteristic of the sphere  $F$ .  $\square$

The quotient space

$$\mathcal{M} := \widetilde{\mathcal{M}} \times_{\text{Aut}(\mathbb{CP}^1)} \mathbb{CP}^1$$

of  $\widetilde{\mathcal{M}} \times \mathbb{CP}^1$  under the diagonal action of the 6-dimensional automorphism group  $\text{Aut}(\mathbb{CP}^1) = \text{PGL}(2, \mathbb{C})$ , where  $\phi \in \text{Aut}(\mathbb{CP}^1)$  acts by

$$(u, z) \mapsto (u \circ \phi^{-1}, \phi(z)),$$

is then a  $2n$ -dimensional manifold, since the spheres in  $\widetilde{\mathcal{M}}$  being simple implies that this action is free. Furthermore, there is a well-defined evaluation map

$$\begin{aligned} \text{ev}: \quad \mathcal{M} &\longrightarrow \widetilde{W} \\ [u, z] &\longmapsto u(z), \end{aligned}$$

where  $[u, z]$  denotes the class represented by  $(u, z)$ .

**6.2. Spheres intersecting an arc.** Let  $\gamma$  be a proper embedding of the interval  $[0, 1]$  or  $[0, 1)$  into  $\widetilde{W}$ , with  $\gamma(0) \in F$  and no other image point of  $\gamma$  in  $\partial P \times \mathbb{CP}^1$ . So in the case of the closed interval  $[0, 1]$  we must have  $\gamma(1) \in M_+$ ; in the case of the half-open interval  $[0, 1)$  we go to  $-\infty$  in the cylindrical end  $(-\infty, 0] \times M$  as we approach 1. Set

$$\mathcal{M}_\gamma := \text{ev}^{-1}(\gamma),$$

where by slight abuse of notation we identify  $\gamma$  with its image in  $\widetilde{W}$ .

**Proposition 6.2.** *Given  $\gamma$ , a generic choice of  $J$  can be made such that  $\mathcal{M}_\gamma$  is a 1-dimensional manifold including one component diffeomorphic to a half-open interval. In particular,  $\mathcal{M}_\gamma$  is not compact.*

*Proof.* Under the identification of  $F = \{*\} \times \mathbb{CP}^1$  with  $\mathbb{CP}^1$ , the space  $\mathcal{M}_\gamma$  contains the class  $[\text{id}_{\mathbb{CP}^1}, \gamma(0)]$ , so  $\mathcal{M}_\gamma$  is non-empty.

For  $\mathcal{M}_\gamma$  to be a 1-dimensional manifold (away from potential boundary points), we need to ensure that the evaluation map  $\text{ev}$  be transverse to the submanifold  $\gamma$  of  $\widetilde{W}$ . By [26, Theorem 3.4.1 and Remark 3.4.8], for generic  $J$  transversality holds for all simple spheres not contained entirely in a region where  $J$  has been fixed by one of the conditions (J1) to (J3). This is the generic choice we want to make in (J4). The aforementioned theorem from [26] only treats the case without boundary, but it can still be applied here. Indeed, for the interval  $[0, 1]$  we have  $\gamma(1) \in M_+$ , where by Lemma 5.2 (iv) there are no non-constant holomorphic spheres. A neighbourhood of the boundary point  $\gamma(0)$  in  $\gamma$  lies in  $U_\partial$ , where the explicit description of holomorphic spheres in the class  $[F]$  as maps of the form  $z \mapsto (p, \phi(z))$  with  $\phi \in \text{Aut}(\mathbb{CP}^1)$  shows that at any point  $[u, z] \in \mathcal{M}$  with  $u(z) \in U_\partial$  the evaluation map  $\text{ev}: \mathcal{M} \rightarrow \widetilde{W}$  is submersive.

For  $(p, w) \in U_\partial \subset P \times \mathbb{CP}^1$ , the preimage  $\text{ev}^{-1}(p, w)$  must be of the form  $[(p, \phi), \phi^{-1}(w)]$ , where  $(p, \phi)$  denotes the holomorphic sphere  $z \mapsto (p, \phi(z))$ , so this preimage consists in fact of the single point  $[(p, \text{id}_{\mathbb{CP}^1}), w]$ .

So near the preimage of  $\gamma(0)$ , the moduli space  $\mathcal{M}_\gamma$  looks like a single half-open interval, and there are no other boundary points in  $\mathcal{M}_\gamma$ . We conclude that the corresponding component is a half-open interval.  $\square$

**6.3. Stable maps.** By Proposition 6.2 we can find a sequence in  $\mathcal{M}_\gamma$  without any convergent subsequence. We now want to show that such a sequence cannot have any subsequence Gromov-converging to a stable map in the sense of [26, Definition 5.1.1]. Together with the compactness theorem from symplectic field theory [5] this will imply, for a generic choice of contact form  $\alpha$ , that there has to be a Gromov–Hofer-convergent subsequence where breaking occurs. This will lead to the existence of periodic Reeb orbits.

Naively speaking, the non-existence of a Gromov-convergent subsequence follows from the fact that bubbling is a phenomenon in codimension at least 2 (in the  $\pi$ -semipositive situation) — see the dimension formula for the moduli space

$\mathcal{M}_{T'}^*(\{A_\beta\})$  in the proof of Lemma 6.3 below — and we are only considering a 1-parameter family of holomorphic curves.

For the formal argument, suppose  $[u_\nu, z_\nu]$  is a sequence in  $\mathcal{M}_\gamma$  with  $(u_\nu, z_\nu)$  Gromov-convergent to a stable map  $(\{u_\alpha\}_{\alpha \in T}, z)$  modelled on a tree  $T$ . Write  $e(T)$  for the number of edges of  $T$ . We want to show  $e(T) = 0$ , in which case the limit would be a classical one in the  $C^\infty$ -topology.

By [26, Proposition 6.1.2] we find a simple stable map  $(\{v_\beta\}_{\beta \in T'}, z')$ , i.e. with each non-constant  $v_\beta$  a simple map and with different non-constant spheres having distinct images, such that

$$\bigcup_{\beta \in T'} v_\beta(\mathbb{CP}^1) = \bigcup_{\alpha \in T} u_\alpha(\mathbb{CP}^1).$$

Moreover, with  $A_\beta \in H_2(\widetilde{W})$  denoting the homology class represented by the holomorphic sphere  $v_\beta$ , and with suitable weights  $m_\beta \in \mathbb{N}$ , we have

$$[F] = \sum_{\beta \in T'} m_\beta A_\beta.$$

There are two distinguished vertices in the bubble tree  $T'$ ; these may coincide. One is the bubble  $v_{\beta_0}$  corresponding to the limit marked point  $z'$ , in particular  $v_{\beta_0}(z') \in \gamma$ . Notice that  $v_{\beta_0}$  may well be constant, a so-called ghost bubble. This is the case if the image of  $z'$  on  $\gamma$  happens to be the image of a nodal point joining two non-constant bubbles.

The second is the bubble  $v_{\beta_\infty}$  with  $v_{\beta_\infty}(\mathbb{CP}^1) \cap C_\infty \neq \emptyset$ . We claim that this property uniquely determines  $v_{\beta_\infty}$ , and  $m_{\beta_\infty} = 1$ . The homological intersection of  $F$  with  $P_\infty$  equals 1, so there has to be at least one non-constant sphere  $v_\beta$  intersecting  $P_\infty$ . There are no non-constant holomorphic spheres contained in  $P_\infty$  (thanks to the strictly plurisubharmonic function  $\psi_P$  and the maximum principle), so positivity of intersection with the holomorphic hypersurface  $P_\infty$ , see [9, Proposition 7.1], tells us that there is a unique  $v_\beta$  intersecting  $P_\infty$ , and this sphere has to be simple. From Lemma 5.2 it then follows that

$$v_\beta(\mathbb{CP}^1) \subset \widetilde{W} \setminus C_\infty \text{ for } \beta \neq \beta_\infty.$$

**Lemma 6.3.** *The tree  $T'$  consists of a single vertex, i.e.  $e(T') = 0$ .*

*Proof.* The symplectic energy  $\int_{\mathbb{CP}^1} u^* \tilde{\omega} = \tilde{\omega}([u])$  of any sphere  $u \in \widetilde{\mathcal{M}}$  equals  $\omega_{\text{FS}}([F]) = \pi$ . Assuming  $e(T') \geq 1$  (which implies that there are at least two non-constant bubbles), we have  $0 < \omega(A_\beta) < \pi$  for every  $\beta \in T'$  with  $A_\beta \neq 0$ . For  $\beta \neq \beta_\infty$  we may regard  $A_\beta$  for homological computations as a spherical class in  $H_2(W)$ . Our choice (J4) of almost complex structure was made so as to guarantee regularity for spheres. Then the moduli space of simple holomorphic spheres in any class  $A_\beta \neq 0$  is a non-empty manifold of dimension  $2n + 2c_1(A_\beta) - 6$ , which implies  $c_1(A_\beta) \geq 3 - n$ . So the requirement that  $(W, \omega)$  be  $\pi$ -semipositive implies

$$c_1(A_\beta) \geq 0 \text{ for } \beta \neq \beta_\infty.$$

As in the proof of Proposition 6.2, by possibly refining the generic choice of almost complex structure in (J4) we can ensure that the moduli space  $\mathcal{M}_{T'}^*(\{A_\beta\})$  of unparametrised simple stable maps modelled on  $T'$  (with a single marked point) is a smooth manifold. As shown in [26, Theorem 6.2.6], the dimension of this moduli

space equals

$$\dim \mathcal{M}_{T'}^*(\{A_\beta\}) = 2n + 2c_1\left(\sum_{\beta} A_\beta\right) - 4 - 2e(T').$$

Now consider the evaluation map

$$\begin{aligned} \text{ev}_0: \quad \mathcal{M}_{T'}^*(\{A_\beta\}) &\longrightarrow \widetilde{W} \\ [\{v_\beta\}, z'] &\longmapsto v_{\beta_0}(z'). \end{aligned}$$

Since transversality of the evaluation map is automatic at ghost spheres, we see as in Proposition 6.2 that  $\text{ev}_0^{-1}(\gamma)$  is a non-empty manifold of dimension

$$\begin{aligned} \dim \text{ev}_0^{-1}(\gamma) &= \dim \mathcal{M}_{T'}^*(\{A_\beta\}) - (2n - 1) \\ &= 2c_1\left(\sum_{\beta} A_\beta\right) - 2e(T') - 3 \\ &= 2c_1(A_{\beta_\infty}) + 2c_1\left(\sum_{\beta \neq \beta_\infty} A_\beta\right) - 2e(T') - 3 \\ &\leq 2c_1(A_{\beta_\infty}) + 2c_1\left(\sum_{\beta \neq \beta_\infty} m_\beta A_\beta\right) - 2e(T') - 3 \\ &= 2c_1([F]) - 2e(T') - 3 \\ &= 1 - 2e(T'). \end{aligned}$$

This dimension must be non-negative, hence  $e(T') = 0$  after all.  $\square$

One of the reasons why a stable map may fail to be simple is that it might contain several components having the same image. In this case  $e(T)$  would be larger than  $e(T')$ . In our situation, however, an intersection argument with the class  $[P_\infty]$  as in the proof of Proposition 6.2 allows us to conclude that  $T$ , likewise, must consist of a single vertex  $\alpha_\infty$ , with  $u_{\alpha_\infty}$  a simple map.

**6.4. Proof of the main theorem.** As shown in the final paragraph of [1], by invoking the Arzelà–Ascoli theorem one can reduce the proof of Theorem 3.1 to the case where in addition the contact form  $\alpha$  is assumed to be non-degenerate (i.e. where the linearised Poincaré return map along closed orbits of the Reeb vector field  $R_\alpha$ , including multiples, does not have an eigenvalue 1). Under this assumption, the compactness theorem from symplectic field theory [5, Theorem 10.2] applies. This tells us that we can find a sequence in  $\mathcal{M}_\gamma$  convergent (in the sense of that paper) to a holomorphic building of height  $k_-|1$  with  $k_- > 0$ . The lowest level of this building consists of holomorphic curves in the symplectisation  $(\mathbb{R} \times M, d(e^s \alpha))$  with positive punctures only and symplectic energy smaller than  $\pi$ , for the total symplectic energy of the holomorphic building, which is a homological invariant, equals  $\pi$ . So these positive punctures constitute a null-homologous Reeb link in  $(M, \alpha)$  of total action smaller than  $\pi$ .

Any holomorphic building of genus 0 and height  $k_-|1$  with  $k_- > 0$  contains at least two finite energy planes. By the argument in the proof of Lemma 5.2, at most one of these can intersect  $P_\infty$ ; any other finite energy plane in the top level  $W \cup_S C_\infty$  of the building stays inside  $W$ . In the Liouville case there can be no finite energy plane in  $W$  with a negative puncture, because by Stokes’s theorem its energy would be negative; cf. [5, Lemma 5.16]. Hence, in the Liouville case there has to be a finite energy plane with a positive puncture in one of the lower



levels  $(\mathbb{R} \times M, d(e^s \alpha))$  of the building, corresponding to a contractible Reeb orbit of period smaller than  $\pi$ . This concludes the proof of Theorem 3.1.

**6.5. Proof of Theorem 3.4.** Arguing by contradiction, we assume that  $M_+$  is not empty. Then we can choose a path  $\gamma$  in  $W$  from  $F$  to  $M_+$  as in Section 6.2. Since the concave end  $M$  of  $W$  is assumed to be empty, the non-compactness of  $\mathcal{M}_\gamma$  can only be caused by bubbling, which is precluded by the argument in Section 6.3. In other words,  $\mathcal{M}_\gamma$  would have to be compact, contradicting Proposition 6.2.

**6.6. Relation with the work of Oancea–Viterbo.** The recent paper of Oancea–Viterbo [30] also grew from taking a fresh look at the work of McDuff [25]. Their main interest is the topology of symplectic fillings. Although their approach seems quite different on a technical level, one can interpret their results in our set-up as statements about the special case where  $M$  is empty (and hence so is  $M_+$ , by Theorem 3.4).

In that special case, one can define the moduli space  $\mathcal{M}_{-1,1,\infty}$  of holomorphic spheres  $u \in \widetilde{\mathcal{M}}$  with  $u(z) \in P \times \{z\}$  for  $z \in \{-1, 1, \infty\}$ . Requiring the image of three marked points to lie on three respective hypersurfaces is essentially equivalent to taking the quotient under the action of the automorphism group  $\text{Aut}(\mathbb{CP}^1)$ . For generic  $J$  subject to conditions (J1) to (J4) the moduli space  $\mathcal{M}_{-1,1,\infty}$  turns out to be a *compact* oriented manifold (with boundary) of dimension

$$2n + 2c_1([F]) - 3 \cdot 2 = 2n - 2.$$

Since the holomorphic spheres near the boundary  $\partial P \times \mathbb{CP}^1$  of  $\widetilde{W}$  are the obvious ones by Lemma 5.2, one sees that the evaluation map

$$\text{ev}: \mathcal{M}_{-1,1,\infty} \times \mathbb{CP}^1 \rightarrow \widetilde{W},$$

which maps boundary to boundary, is of degree 1. The homological arguments in the proof of [30, Proposition 2.11] go through unchanged, even though in our set-up we work with manifolds with boundary. In this way one can derive one of their main theorems [30, Theorem 2.6]. The homological conditions imposed there are necessary, in general, to guarantee the compactness of the moduli space. Let us point out one special case of their theorem where these homological conditions are superfluous.

**Theorem 6.4** (Oancea–Viterbo). *Let  $S$  be as in Section 3.1 and  $(W, \omega)$  a symplectically aspherical strong symplectic filling of  $(S, \ker \alpha_S)$ . Then the homomorphism  $H_j(S; \mathbb{F}) \rightarrow H_j(W; \mathbb{F})$  on homology with coefficients in any field  $\mathbb{F}$ , induced by the inclusion  $S \rightarrow W$ , is surjective in all degrees.*  $\square$

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